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From canonical tensor operators of SU(3) and $u_q(3)$ to bi-orthogonal coupling coefficients: explicit expansion

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Abstract. Expansion of the matrix elements of SU(3) and $u_q(3)$ canonical tensor operators in terms of the bi-orthogonal coupling coefficients and overlaps of the Draayer–Akiyama construction are considered. Special bi-orthogonal extremal isoscalar factors (with subscripts as multiplicity labels and proportional to q-Racah (q-$6j$) coefficients or the generalized Wilson–Rahman rational bi-orthogonal functions in terms of balanced ${}_4F_3(1)$ or ${}_4\phi_3(q)$ hypergeometric series) are used as a natural basis for extremal matrix elements of the highest weight component of the canonical tensor operators of SU(3) in the generating function approach of Biedenharn, Lohe and Louck. The expansion that is obtained (triple sum), together with previously derived asymmetric seed isofactors and elementary overlaps, gives the explicit overlap coefficients and can be used to derive SU(3) and $u_q(3)$ canonical tensor operators as well as new explicit normalized seed isofactors with Regge-type symmetry, specified for the minimal null space case.

1. Introduction

The importance of coupling coefficients and irreducible tensor operators of unitary groups for physical applications is well known. Many analytical solutions of the coupling problem for the SU(3) and SU(n) groups (including multiply occurring irreducible representations (irreps) in the coproduct decomposition) with non-orthogonal coupling (Clebsch–Gordan or Wigner) coefficients have been proposed. These results have been derived by means of an integration over the group [1], projection operator methods [2–4], and recursive-recoupling techniques [5–12] (including the construction of explicit bi-orthogonal systems [9–11]) or by using different generating invariants [13–15] as well as vector-coherent-states techniques [16]. Some of the methods have been extended to the quantum groups $u_q(n)$ [17], particularly to $u_q(3)$ [18], with the bi-orthogonal coupling coefficients (isoscalar factors) of the quantum groups expressed as sums involving multiplicity free coupling coefficients [19].

Nevertheless, the SU(3) canonical tensor operator concept of Biedenharn, Louck and their collaborators [20–23] has not lost its attractiveness, especially for numerical applications [24]. However, the rather complicated generating and denominator function technique of this approach [22, 23] cannot be easily extended from SU(3) to $u_q(3)$, and explicit analytical expressions are still only available for the case of a canonical tensor operator of rank (1 1) [25] and for some extremal matrix elements of tensor operators (with maximal or minimal null spaces) and the corresponding orthonormal canonical isoscalar factors (isofactors) of SU(3) [26] and $u_q(3)$ [18, 27]. Unfortunately, a conjecture

[26] concerning the possibility of a straightforward construction of the orthogonal SU(3) canonical tensor operators for non-extremal values of the multiplicity label has not been confirmed [27], so the general orthonormalization problem for the Draayer–Akiyama [24] construction remains unsolved analytically.

The concept of bi-orthogonal systems (Ališauskas [11, 17, 18]) not only leads to analytical expressions with minimal total number of summation parameters (which in the SU(3) or $u_q(3)$ case never exceeds 6) but also brings a systematic approach to the coupling problem and allows one to reduce the global coupling coefficients to their expansion [10, 11, 19, 28] in terms of specific boundary (seed) isofactors (cf also [4, 16, 29]) in analogy with the construction of stretched $SU(3) \supset SO(3)$ basis states [30]. Ališauskas [31, 32] presented the boundary (seed) isofactors (with highest-weight or lowest-weight values of definite states) for orthonormal (paracanonical and pseudocanonical) couplings in SU(3). The corresponding seed isofactors of $u_q(3)$ can be derived by means of a self-consistent Gram–Schmidt procedure in terms of Gram determinants (cf [11, 12]) formed by overlaps [18] of the coupled states. (These overlaps are equivalent to some bilinear combinations of isofactors. Note that in [12] a similar procedure for overlaps, equivalent to some SU(3) seed recoupling coefficients, is used.)

Definite symmetric and asymmetric seed isofactors are expressed, for the maximal null space of the canonical tensor operators of $u_q(3)$, as single sums in section 5 of [18] (cf [33] for a triple-sum formulation) when the asymmetric seed isofactors are extended to a double sum [18] for the general case of the non-orthonormalized [24, 26] construction. Although the structure of symmetric seed isofactors was predicted for the maximal null space of the canonical tensor operators of SU(3) as (20) of [24], the correlation of their structure with defining relations of the canonical tensor operators [22] in the generic multiplicity label case is not transparent. Indeed the compositions of two other types of seed isofactors, (3.9) or (3.12) of [27] (with the maximal isospin states appearing in different positions and hence correlated with canonical splitting condition [22]), were used for the elimination of indefinite $0/0$ solutions of the normalization problem (cf (3.7) and (3.14) of [27]) for the canonical SU(3) and $u_q(3)$ isofactors with maximal null space. However, a general solution of the system of equations given in (2.13) of [27] is only possible under specific restrictions (see section 4 of [27] where some inconsistencies of numerically distinctive conditions are discussed) and is not directly correlated with the general null space approach to canonical tensor operators [20–23] induced by a polynomial structure of their reduced matrix elements.

In fact, overlaps for the Draayer–Akiyama [24, 26] construction for $u_q(3)$ can be expanded by means of asymmetric seed isofactors, (5.18) of [18], in terms of overlaps, (3.24) of [18], of dual coupled superscript states, using some symmetry properties of overlaps if necessary. Unfortunately, the analytic regions of these functions are usually mutually exclusive, although under specific restrictions the overlaps, (3.6) of [11], in terms of a (non-very) well-poised series (or their q-version [17]) may be more usable. Unlike the case of dual coupled states (for which the analyticity is interrupted by the appearance of superfluous states), the overlaps of the coupled subscript states are analytical in all regions of the parameter space, but the labels for expansions in terms of subscript states are not simply correlated with the values of the usual seed isofactors. Furthermore, some corresponding elementary expansion coefficients have been derived [18] in different regions by different methods.

In this paper the approach of Biedenharn *et al* [22] is extended to an alternative class of boundary isofactors for SU(3) and $u_q(3)$. Recall that Biedenharn *et al* used (as their main tool for constructing the complete set of the denominator functions) the distinctive polynomial properties of the numerator function of the extremal reduced matrix element of the highest-weight component of the unit SU(3) canonical tensor operator (projective tensor operator (2.5a) of [22] or isofactors)

$$\langle aby\, i' + i_0''||T^{(a''b'')t}_{y_0''i_0''}||a'b'y'i'\rangle = (a'b'y'i'; a''b''y_0''i_0''||t; aby\, i' + i_0'') \qquad (1.1)$$

of the multiplicity $\mathcal{M}$ with canonical multiplicity label t. In (1.1), numerator polynomials of degree $\mathcal{M}-t$ in terms of $i'+z'$ appear which are independent of $i'-z'$. Here and in what follows we use the same notation for irreps and basis states of SU(3) and $u_q(3)$ as was used in [11, 18, 26, 27] with (ab) for mixed tensor irreps, $a = m_{13} - m_{23}$, $b = m_{23} - m_{33}$ where $[m_{13}, m_{23}, m_{33}]$ is a partition and m_{ij} are the Gelfand–Tsetlin parameters. The basis states are labelled by the hypercharge $y = m_{12}+m_{22}-\frac{2}{3}(m_{13}+m_{23}+m_{33})$ (or $z = \frac{1}{3}(b-a)-\frac{1}{2}y$), the isospin $i = \frac{1}{2}(m_{12} - m_{22})$ and its projection $i_z = m_{11} - \frac{1}{2}(m_{12} + m_{22})$. The parameter z (or $j = \frac{1}{2}b - z$ [3, 16]) is frequently more convenient in explicit expressions than y, because linear combinations $i \pm z \geqslant 0$, $a + z - i \geqslant 0$, $b - z - i \geqslant 0$ are integers and in many situations we can avoid inconvenient fractions. For the state with irrep (ab) in the coproduct $(a'b')\otimes(a''b'')$ decomposition, $z = z'+z''+v$, where again $v = \frac{1}{3}(a'-b'+a''-b''-a+b)$ is an integer. The parameters of the highest weight state (HWS) take on the values $y_0 = \frac{1}{3}(a+2b)$, $i_0 = \frac{1}{2}a = -z_0$, while for the lowest weight state (LWS) $\overline{y}_0 = -\frac{1}{3}(2a + b)$, $\overline{i}_0 = \frac{1}{2}b = \overline{z}_0$ and for the maximal isospin state (MIS) $y_{\mathrm{m}} = \frac{1}{3}(a - b)$, $i_{\mathrm{m}} = \frac{1}{2}(a + b)$, $z_{\mathrm{m}} = \frac{1}{2}(b - a)$. The multiplicity $\mathcal{M}$ of the tensor operators $T^{a''b''t}_{y_0''i_0''i_z''}$ with fixed shifts $a - a'$ and $b - b'$ may exceed the multiplicity r of irrep (ab) in the $(a'b') \otimes (a''b'')$ decomposition (see [18, 27]) and the lowest values of the canonical multiplicity label t may be eliminated by the null space inclusion property [22].

We doubt whether the cumbersome generating function technique of [22] based on polynomial properties of (1.1) can be extended straightforwardly to $u_q(3)$. Nevertheless, the modified [26] Draayer–Akiyama [24] construction and explicit expressions of the extremal canonical isofactors [18, 27] insures an analytically distinctive q-polynomial structure for the reduced matrix elements

$$\langle aby\, i' + i_0''||\tilde{T}^{(a''b'')t,q}_{y_0''i_0''}||a'b'y'i'\rangle_q \qquad (1.2)$$

of the $u_q(3)$ twisted [34] tensor operators

$$\tilde{T}^{(a''b'')t=k+1,q}_{y''i''i_z''} = [T^{(kk)t,q}\, T^{(a''-k,b''-k)1,q}]^{(a''b'')q}_{y''i''i_z''} \qquad (1.3)$$

which are derived by means of the stretched [19] coupling of the auxillary canonical tensor operators. Operator $T^{(a''-k,b''-k)1,q}_{y_1j_1m_1}$ with maximal null space in (1.3) insures the full shifts of the $u_q(3)$ irrep parameters and the null space inclusion property of the $u_q(3)$ canonical tensor operators (after eliminating the superfluous tensor operators by means of the orthogonalization process of (1.3) starting from the maximum value of t), when the self-adjoint canonical tensor operators $T^{(kk)t,q}_{y_2j_2m_2}$ have minimal null space [18] and give trivial (zero) shift of the $u_q(3)$ irrep parameters, as well as zero shift of the $u_q(2)$ irreps for maximal value of $j_2 = k = t - 1$.

In this paper we exploit the rather simple algebraic structure of extremal isofactors, related to the right-hand side of (1.2), but with a multiplicity labelling of a different kind. In section 2 a natural basis for extremal matrix elements in terms of the bi-orthogonal extremal isofactors $(a'b'y'i'; a''b''y_0''i_0''||_{+,\tilde{j}'',+}; aby\, i' + i_0'')_q$ with subscript is proposed. (Here and in what follows the $+$ and $-$ signs and their positions in the multiplicity labels (subscripts) $+, \tilde{j}'', +$; $-, +, \tilde{J}$ or $\tilde{I}', -, -$ indicate the signs and positions of the extremal z', z or z'', that is, the LWS and HWS in the corresponding isofactor.) These special isofactors are also proportional to definite q-Racah (q-6j) coefficients as well as to the generalized Wilson–Rahman [35] rational bi-orthogonal functions in terms of balanced ${}_4F_3(1)$ or ${}_4\phi_3(q)$

basic hypergeometric series. Hence the expansion may be inverted. In section 3 the composition of (1.2) and the above-mentioned inverse expansion is considered. After a rearrangement of the single and multiple sums we find an explicit expansion of the matrix elements $\langle aby\, i' + i''_0 || \tilde{T}^{(a''b'')t,q}_{y''_0 i''_0} || a'b'y'i' \rangle_q$ by means of a triple sum in terms of isofactors $(a'b'y'i'; a''b''y''_0 i''_0 ||_{+,\tilde{J}'',+}; aby\, i' + i''_0)_q$. Finally, overlap coefficients for the non-orthonormal tensor operators $\tilde{T}^{(a''b'')t=k+1,q}_{y''i''i''_z}$, which are $\mathrm{u}_q(3)$ or SU(3) unit canonical tensor operators $T^{(k\,k)t,q}_{y_2 j_2 m_2}$ after a Gram–Schmidt procedure, are given. In section 4, these results are specified for explicit normalized boundary isofactors with the multiplicity label corresponding to the minimal null space case. These extremal seed isofactors are presented in terms of ${}_3\phi_2(q)$ or ${}_3F_2(1)$ series, with the normalization function in terms of the double sum related to a q-extension of the denominator polynomial [23] of the SU(3) canonical tensor operators.

2. Rahman's bi-orthogonal functions and alternative expansion of bi-orthogonal coupling coefficients

We use the Cartan–Weyl generators E_{ik} $(i, j, k = 1, 2, 3)$ of the unitary quantum algebra $\mathrm{u}_q(3) = U_q(u(3))$, with generic q and composite generators expressed in terms of q-deformed commutators, which satisfy the commutation relations [19, 27] and the corresponding coproduct expansion rules. Here and in what follows $[x]$ and $[x]!$ are, respectively, the q-numbers and q-factorials,

$$[x] = (q^x - q^{-x})/(q - q^{-1}) \qquad [x]! = [x][x-1]\ldots[2][1]$$
$$(\alpha|q)_n = \prod_{k=0}^{n-1}[\alpha + k] \qquad [1]! = [0]! = (\alpha|q)_0 = 1 \tag{2.1}$$

which are invariant under substitution $q \leftrightarrow q^{-1}$.

Perhaps the simplest expression (without sums) is for the isofactors

$$(a'b'y'i'; a''b''y''_0 i''_0 ||\rho; aby\, i' + i''_0)_q$$

with multiplicity label of the superscript $\rho \equiv -, +, \tilde{J}$ type. These particular coefficients can be derived (applying isofactor symmetry) by means of (4.6) of [17]. The expansion problem in terms of such superscript isofactors is solved [18] by the usual seed isofactor technique. For an alternative expansion, we first rearrange the bi-orthogonal isofactors

$$\begin{aligned}(a'b'y'i'; a''b''y''\, i' + \tfrac{1}{2}a ||_{-,+,\tilde{J}}; aby_0 i_0)_q \\ = \sum_{\tilde{I}'} (\eta_{-,+,\tilde{J}} | \eta_{\tilde{I}',-,-})_q (a'b'y'i'; a''b''y''\, i' + \tfrac{1}{2}a ||^{\tilde{I}',-,-}; aby_0 i_0)_q\end{aligned} \tag{2.2}$$

expanding the right-hand side isofactor (expressed in terms of stretched q-$6j$ coefficients—(4.6) of [17]) by means of the overlap matrix $(\eta_{-,+,\tilde{J}} | \eta_{\tilde{I}',-,-})_q$ (given in terms of a balanced ${}_5F_4(q)$ hypergeometric series—(3.7) of [18])†. Using a new version (cf (3.6) of [18]) of the

† Note that in (3.21*a*) of [18] the signs of A'_1, A'_2 and A'_3 in the upper entries of ${}_5F_4(q, 1)$ should be the opposite of what is given; the definitions of A'_3 in (3.21*b*) and p'_5 in (3.25) should be corrected (by +1 and −1, respectively); the bottom entry $-N_1 - B$ of ${}_5F_4$ in (3.22*b*) should be changed to $-N_1 - B + 1$; the first factors in (4.15*a*) and (4.15*b*) should be replaced by $q^{(a-b)(k+2)}$; parameter v should be omitted on the right-hand side of (5.5); an additional phase factor $(-1)^{a'+a''-a+v}$ should be included on the right-hand side of (5.11) and appear instead of $(-1)^v$ on the right-hand side of (5.14) where the q-exponent parameter a'' also should be replaced by a; $(a' + a'' + a - v)$ should be replaced by $(a' + a'' - a - v)$ in (5.13) and (5.16), with the opposite signs of the terms bv and $\tilde{I}''(\tilde{I}'' + 1)$ in (5.16).

summation formula (cf (2.4.2) of [36]) of a special very well-poised basic hypergeometric series ${}_6\phi_5$ (or ${}_5F_4$),

$$\sum_j \frac{[2j+1][j-p_1-1]!(-1)^{p_1+j+1}}{[p_1+j+1]![p_2-j]![p_2+j+1]![p_3-j]![p_3+j+1]![p_4-j]![p_4+j+1]!} = \frac{[p_1+p_2+p_3+p_4+2]!}{\prod_{1\leqslant k<n\leqslant 4}[p_k+p_n+1]!} \tag{2.3}$$

we obtain an expression for (2.2) as a single sum in terms of a balanced ${}_4F_3(q)$ or ${}_4\phi_3$ hypergeometric series. After using the isofactor symmetries, a natural basis for extremal $u_q(3)$ and SU(3) tensor operators can be given in terms of the biorthogonal isofactors with subscript $+, \tilde{j}'', +$ correlated with the LWS of the first and resulting irreps:

$$\begin{aligned}
(\zeta_{i'+z'}|\eta_{+,\tilde{j}'',+})_q &= (a'b'y'i'; a''b''y_0''i_0''||_{+,\tilde{j}'',+}; aby\, i'+i_0'')_q \\
&= \sum_t (a'b'\overline{y}_0'\overline{i}_0'; a''b''\tilde{y}''\tilde{j}''||t; ab\overline{y}_0\, \overline{i}_0)_q (a'b'y'i'; a''b''y_0''i_0''||t; aby\, i'+i_0'')_q \\
&= \frac{([b-z-i]![b-z+i+1]![i+z]![i-z]!)^{1/2}}{[a'+a''-v+1]!\nabla[\frac{1}{2}b'-z', \frac{1}{2}b', i']} \\
&\quad\times \frac{([2\tilde{j}''+1][\tilde{j}''-\tilde{z}'']!)^{1/2} q^{\tilde{Q}+\tilde{R}} W}{([\tilde{j}''+\tilde{z}'']!)^{1/2}\nabla[\frac{1}{2}b, \frac{1}{2}b', \tilde{j}'']H[a''b''\tilde{j}''\tilde{z}'']} \\
&\quad\times \sum_s \frac{(-1)^{v+s}[\tilde{j}''+\tilde{z}''+s]![b'+v-s]![a'+a''+z'+i'-s+1]!}{[s]![\tilde{j}''-\tilde{z}''-s]![-v+s]![i+z-s]![a'+a''+b-v-s+2]!}
\end{aligned} \tag{2.4}$$

where $\tilde{z}'' = \frac{1}{2}(b-b') - v$, $i+z = i'+z'+v$,

$$\tilde{z}' = \frac{1}{2}(a''-a) - v \qquad \tilde{z} = \frac{1}{2}(b''-a') + v)$$

and

$$\begin{aligned}
W &= [a+1][a+b+2]([b+1][a]![a+b+1]![a']![a'+b'+1]!)^{1/2} \\
&\quad\times \left(\frac{[2i'+1]![a'+z'-i']![a'']![b'']![a''+b''+1]!}{[2i+1]![a'+z'+i'+1]![a+z-i]![a+z+i+1]!}\right)^{1/2}
\end{aligned} \tag{2.5}$$

$$\nabla[abc] = \left(\frac{[a+b-c]![a-b+c]![a+b+c+1]!}{[b+c-a]!}\right)^{1/2} \tag{2.6}$$

$$H[abiz] = ([a+z-i]![a+z+i+1]![b-z-i]![b-z+i+1]!)^{1/2} \tag{2.7}$$

$$\begin{aligned}
\tilde{Q} &= Q_1(b'a'aba''b''; \tilde{j}''\tilde{z}'') + \tfrac{1}{2}\tilde{z}'(3\tilde{z}'+2a'-2b') + \tfrac{1}{8}(a^2-a''^2) - \tfrac{1}{2}a'b' - b' \\
&\quad + \tfrac{1}{2}(a+b-3a'-a''-b'')
\end{aligned} \tag{2.8}$$

$$\tilde{R} = \tfrac{1}{2}(a+2z)(\tfrac{1}{2}a''+b'') - \tfrac{1}{2}a''i' \tag{2.9}$$

$$\begin{aligned}
Q_1(a'b'a''b''a\, b; \tilde{J}\tilde{z}) &= Q_1(b''a''b'a'b\, a; \tilde{J}, -\tilde{z}) = \tfrac{1}{2}\{\tilde{J}(\tilde{J}+1) \\
&\quad + \tilde{z}(3\tilde{z}+2a-2b) - ab + \tfrac{1}{2}(a'+b'') + a'' + b' - a - b\}.
\end{aligned} \tag{2.10}$$

Using the notation [37]

$${}_{p+1}F_p\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_{p+1} \\ \beta_1, \ldots, \beta_p\end{matrix}; q, x\right] = \sum_k^{\infty} \frac{(\alpha_1|q)_k(\alpha_2|q)_k \ldots (\alpha_{p+1}|q)_k}{(\beta_1|q)_k \ldots (\beta_p|q)_k (q|q)_k} x^k \tag{2.11}$$

(with $x = q^{\pm(c+1)}$, $c = -1$ for the balanced series) instead of the standard definition [36]

$$ {}_{p+1}\phi_p \left[\begin{matrix} q^{\alpha_1}, q^{\alpha_2}, \ldots, q^{\alpha_{p+1}} \\ q^{\beta_1}, \ldots, q^{\beta_p} \end{matrix} ; q, z \right] = {}_{p+1}F_p \left[\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_{p+1} \\ \beta_1, \ldots, \beta_p \end{matrix} ; q^{1/2}, q^{(c-1)/2} z \right] \tag{2.12} $$

we see that the sum on the right-hand side of (2.4) corresponds to a balanced ${}_4\phi_3$ basic (or ${}_4F_3(1)$ classical) hypergeometric series and is proportional to the balanced series

$$ {}_4F_3 \left[\begin{matrix} -\tilde{j}'' + \tilde{z}'', \tilde{j}'' + \tilde{z}'' + 1, -i - z, -a' - a'' - b + v - 2 \\ -b' - v, -v + 1, -a' - a'' - i' - z' - 1 \end{matrix} ; q, 1 \right] \tag{2.13} $$

or to the q-6j coefficients

$$ \left\{ \begin{matrix} \frac{1}{2}b' & \frac{1}{2}b & \tilde{j}'' \\ \frac{1}{2}(a' + a'' + i' + z' + 1) & \frac{1}{2}(a + b'' + v + i' + z' + 1) & \frac{1}{2}(a' + a'' + b - i' - z' + 1) - v \end{matrix} \right\}_q \tag{2.14} $$

with $\tilde{j}'' - \tilde{z}''$ and $i' + z'$ appearing instead of the matrix indices of the generalized Wilson–Rahman [35] bi-orthogonal functions $R_n^{(2)}[x]$ and $S_n^{(2)}[x]$ in terms of the balanced (basic) ${}_4F_3$ or ${}_4\phi_3$ hypergeometric functions. We note that the appearance of the second matrix indices on the top and bottom rows of (2.13) is rather unusual for Racah coefficients.

In accordance with [35], the dual function appearing in this case is proportional to the balanced series

$$ {}_4F_3 \left[\begin{matrix} -\tilde{j}'' + \tilde{z}'', \tilde{j}'' + \tilde{z}'' + 1, -b' + z' + i', a + b'' + v + 3 \\ -b' - v, b - b' - v + 1, a + b'' + i + z + 4 \end{matrix} ; q, 1 \right]. \tag{2.15} $$

For proof of the bi-orthogonality in the general q case we may use the relationship

$$ \begin{aligned} &\sum_{i'+z'=i+z-v} \sum_{s,s'} \frac{(-1)^s [\tilde{j}'' + \tilde{z}'' + s]! [b' + v - s]! [a' + a'' + z' + i' - s + 1]! [i' + z']!}{[s]! [\tilde{j}'' - \tilde{z}'' - s]! [-v + s]! [i + z - s]! [a' + a'' + b - v - s + 2]!} \\ &\quad \times \frac{(-1)^{s'} [\tilde{I}'' + \tilde{z}'' + s']! [b' + v - s']! [a + b'' + v + s' + 2]!}{[s']! [\tilde{I}'' - \tilde{z}'' - s']! [b - b' - v + s']! [b' - z' - i' - s']!} \\ &\quad \times \frac{[b - z - i]! [a + b' + b'' + v + 3]!}{[a + b'' + i + z + s' + 3]! [a' + a'' - v + 1]!} \\ &= (-1)^{\tilde{j}'' - \tilde{z}''} \delta_{\tilde{j}'', \tilde{I}''} \frac{[\tilde{j}'' + \tilde{z}'']! [\frac{1}{2}(b + b') - \tilde{j}'']! [\frac{1}{2}(b + b') + \tilde{j}'' + 1]!}{[2\tilde{j}'' + 1] [\tilde{j}'' - \tilde{z}'']!}. \end{aligned} \tag{2.16} $$

The sum over $i' + z' = i + z - v$ is summable in terms of a balanced ${}_3\phi_2$ (or ${}_3F_2(1)$) hypergeometric series, with the remaining sums also turning into a balanced ${}_3\phi_2$ form.

This bi-orthogonality relation may be reformulated for the usual q-6j coefficients [38, 39]:

$$ \begin{aligned} &\sum_x \left\{ \begin{matrix} a & b & e \\ \frac{1}{2}x + k & \frac{1}{2}x & r - \frac{1}{2}x \end{matrix} \right\}_q \left\{ \begin{matrix} a & b & e' \\ \frac{1}{2}(x + 1) + k & \frac{1}{2}(x + 1) & r - \frac{1}{2}(x - 1) \end{matrix} \right\}_q \\ &\quad \times \left(\frac{[2e + 1][2e' + 1][a - b + e]! [b + e' - a]!}{[b + e - a]! [a - b + e']!} \right. \\ &\quad \times \left. \frac{[k - e + x]! [k + e + x + 1]!}{[k - e' + x + 1]! [k + e' + x + 2]!} \right)^{1/2} \\ &= -\delta_{ee'} ([r - a + 1][a + r + 2][r + k - b + 1][b + r + k + 2])^{-1/2} \end{aligned} \tag{2.17} $$

with the dual relation written as follows:

$$\sum_e [2e+1]\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q \begin{Bmatrix} a & b & e \\ d' & c' & f' \end{Bmatrix}_q \left(\frac{[d+c-e]![d+c+e+1]!}{[d'+c'-e']![d'+c'+e+1]!}\right)^{1/2}$$
$$= -\delta_{d,d'-1/2}([c'+f'-a][c'+f'+a+1] \times [d'+f'-b][d'+f'+b+1])^{-1/2} \tag{2.18}$$

where $c-c'=d-d'=f'-f$.

Now using the same notation (2.5)–(2.9), the matrix for the inverse expansion in terms of the coupled states $|\eta_{+,\tilde{j}'',+})_q$ may be expressed as follows:

$$(\eta^{+,\tilde{j}'',+}|\zeta^{i'+z'})_q = (-1)^{(b'-b)/2+\tilde{j}''}[a+b'+b''+v+3]!H[a''b''\tilde{j}''\tilde{z}'']$$
$$\times \frac{([b'-z'-i']![b'-z'+i'+1]![i'+z']![i'-z']![2\tilde{j}''+1][\tilde{j}''-\tilde{z}'']!)^{1/2}}{q^{\tilde{Q}+\tilde{R}}W\nabla[\frac{1}{2}b',]\frac{1}{2}b,\tilde{j}'']\nabla[i,\frac{1}{2}b-z,\frac{1}{2}b]([\tilde{j}''+\tilde{z}'']!)^{1/2}}$$
$$\times \sum_{s'} \frac{(-1)^{s'}[\tilde{j}''+\tilde{z}''+s']![b'+v-s']!}{[s']![\tilde{j}''-\tilde{z}''-s']![b-b'-v+s']![b'-z'-i'-s']!}$$
$$\times \frac{[a+b''+v+s'+2]!}{[a+b''+i+z+s'+3]!} \tag{2.19}$$

with

$$\sum_{i'+z'=i+z-v} (\eta^{+,\tilde{j}'',+}|\zeta^{i'+z'})_q(\zeta_{i'+z'}|\eta_{+,\tilde{j}'',+})_q = \delta_{\tilde{j}'',\tilde{j}''}. \tag{2.20}$$

Since the isofactors with $i-z=i'-z'+a''-v$ and $i+z=i'+z'+v$ are used, we may take either $i'-z'\geqslant 0$ or $i-z\leqslant a$. In both cases, conditions $a+z-i\geqslant 0$ or $i'-z'\geqslant 0$, respectively, imply that the non-vanishing isofactors (2.4) or matrix elements† (1.1) and inverse matrix (2.19) may exist only for $a-a''+v\geqslant 0$. The 'local' bi-orthogonality relation (2.20) in general is more convenient than the global bi-orthogonality relation of the dual isofactors

$$\sum_{y'i'y''i''} (a'b'y'i'a''b''y''i''||^{+,\tilde{j}'',+};abyi)_q(a'b'y'i'a''b''y''i''||_{+,\tilde{j}'',+};abyi)_q = \delta_{\tilde{j}'',\tilde{j}''} \tag{2.21}$$

which may also be used (especially with $y=\bar{y}_0$, $i=\bar{i}_0$, when the first isofactor is proportional to the q-Racah coefficient) for expansion of arbitrary isofactors in terms of isofactors with subscript $+,\tilde{J}'',+$.

Note that the bi-orthogonal classical Wilson–Rahman ${}_4F_3(1)$ series also appeared in the inversion problem for overlaps of the dual projected $\mathrm{SU}(n)\supset\mathrm{SO}(n)$ states [40], where the bi-orthogonality relations

$$\left[\frac{(b-s+1)_2(b+s+2)_2}{(2b-2j+1)(2b-2j+3)}\right]^{1/2}\sum_{k\geqslant 0}\frac{1}{1+\delta_{k0}}\left[\frac{(j+k)!(j-k)!}{(j'+k+1)!(j'-k+1)!}\right]^{1/2}$$
$$\times(j,k;s,-k|b-j,0)(j'+1,k;s,-k|b-j'+1,0)=\delta_{jj'} \tag{2.22}$$

$$\sum_j\left[\frac{(b-s+1)_2(b+s+2)_2(j+k)!(j-k)!}{(2b-2j+1)(2b-2j+3)(j'+k+1)!(j'-k+1)!}\right]^{1/2}$$
$$\times(j,k;s,-k|b-j,0)(j+1,k';s,-k'|b-j+1,0)=\delta_{kk'}(1+\delta_{k0}) \tag{2.23}$$

of special SU(2) Clebsch–Gordan coefficients for $b\geqslant 3s$ were also derived using the relation [41] between special ${}_4F_3(1)$ and ${}_3F_2(1)$ series.

† This situation is not observed in [22].

3. Bi-orthogonal expansion of canonical tensor operators

We will now introduce new expressions for SU(3) and $u_q(3)$ canonical isofactors for arbitrary values of the multiplicity label t. An expansion for these coupling coefficients in terms of bi-orthogonal isofactors with superscript $-, +, \tilde{I}$ is given as (5.18) of [18]

$$(\tilde{T}^k|\eta_{-,+,\tilde{I}})_q \equiv U_3 \left\{ \begin{matrix} (a'b') & (a''-k, \overset{1}{b''}-k) & (a\,b) \\ \overset{k+1}{(k\,k)} & (a\,b)_{-,+,\tilde{I}} & (a''b'') \end{matrix} \right\}_q$$

$$= \sum_{t \geqslant k+1} U_3 \left\{ \begin{matrix} (a'b') & (a''-k, \overset{1}{b''}-k) & (a\,b) \\ \overset{k+1}{(k\,k)} & (a\,b) & \overset{t}{(a''b'')} \end{matrix} \right\}_q \times (a'b'y_0'i_0'; a''b''\overline{y}_0''\overline{i}_0''||t; ab\tilde{y}\tilde{I})_q \tag{3.1a}$$

$$= \frac{([a+1][b+1][a+b+2][b']![a'+b'+1]!)^{1/2}\Gamma[ab\tilde{I}\tilde{z}]\nabla[\overline{i}_0''i_0'\tilde{I}]}{([b'']!)^{1/2}\mathcal{N}\binom{q,k}{a''b''}[a'b';ab]}$$
$$\times \sum_{j,j'} \frac{(-1)^{(b''-a''-k)/2-\tilde{I}+j} q^{Q_6+j(j+1)-j'(j'+1)+3\tilde{I}(\tilde{I}+1)/2}}{\nabla^2[\frac{1}{2}(a''-k), \frac{1}{2}a', j]\nabla^2[\frac{1}{2}(a'+a''-a-k)-v, \frac{1}{2}a, j]}$$
$$\times \frac{[2j+1]\,[2j'+1]}{[b-v+\frac{1}{2}(a'+a''-k)-j+1]!\,[b-v+\frac{1}{2}(a'+a''-k)+j+2]!}$$
$$\times \frac{\nabla^2[\frac{1}{2}(a''+b'')-k, j', j]}{\nabla^2[\frac{1}{2}(b''-k), \frac{1}{2}a', j']\nabla^2[\frac{1}{2}k, j', \tilde{I}]\Gamma^2[abj'\tilde{z}-\frac{1}{2}k]} \tag{3.1b}$$

with two 'braided' sums resembling the very well-poised ${}_9\phi_8$ and ${}_{11}\phi_{10}$ basic hypergeometric series and the corresponding q-phase

$$\begin{aligned} Q_6 = {} & (b'-v+1)(b'+b''-b+v-k) - \tfrac{1}{8}(a'+b''-k)(a'+b''-k+2) \\ & -\tfrac{1}{2}\{(a''-k)(a+b'-a''+k+v) - (b''-k)(b''-b+v-k)\} \\ & -\tfrac{1}{4}(a+b-b'-b''-v+k)(a+b-b'-b''-v+k+2) \\ & -\tfrac{1}{8}k^2 + \tfrac{3}{4}k + \tfrac{1}{2}k\tilde{z} \end{aligned} \tag{3.2}$$

(see (5.19) and (5.8) of [18]). Overlaps (3.1c) correspond to the solution of a boundary value problem which involves a recursive construction (1.3), a recoupling technique analogous with (2.13) and (5.1) of [26], and expansion coefficients of tensor operators $\tilde{T}^{(a''b'')t=k+1,q}_{y''i''i''_z}$. Hence they are part of the special recoupling coefficients (3.1a) (with 'mixed' multiplicity labels) and are equivalent to superpositions of 'seed' isofactors in (3.1b). Here and in what follows the renormalization factor

$$\mathcal{N}\binom{q,k}{a''b''}[a'b';ab] = \frac{\mathcal{D}\binom{q,t=1}{a''-k,b''-k}[a'b';a\,b]\mathcal{D}\binom{q,t=k+1}{k\,k}[a\,b;a\,b]}{[a''-k]![b''-k]!([k]!)^{1/2}} \tag{3.3}$$

is expressed in terms of the denominator functions $\mathcal{D}\binom{q,t}{\cdots}[\ldots]$ [18, 27] of the $u_q(3)$ canonical tensor operators with maximal and minimal null space, respectively. The summation parameters j and j' are restricted as follows:

$$\begin{aligned} & \max(\tfrac{1}{2}(a'-a''+k), a+v-\tfrac{1}{2}(a'+a''-k)) \leqslant j \\ & \qquad \leqslant \min(\tfrac{1}{2}(a'+a''-k), \tfrac{1}{2}(a'+a''-k)-v) \end{aligned} \tag{3.4a}$$

$$\max(\tfrac{1}{2}(a' - b'' + k),\, \tfrac{1}{2}(a' - b'' + k) - v,\ \tilde{I} - \tfrac{1}{2}k) \leqslant j' \leqslant \min(\tfrac{1}{2}(a' + b'' - k),\, \tfrac{1}{2}(b'' - a' - k) + a + v,\, \tilde{I} + \tfrac{1}{2}k). \tag{3.4b}$$

The summation intervals do not exceed

$$\min(a'', a'' - v, a' + a'' - a - v, b' + b'' - b + v) - k$$

for j and

$$\min(b'', b'' + v, a - a' + b'' + v, b - b' + a'' - v) - k$$

for j' (cf the elements of array (1.3*b*) of [27] or (2.8*b*) of [18] which characterize the canonical tensor operators—see (1.5*b*) of [27]).

Since the stretched coupling in this case is trivial, the expansion coefficients of the twisted tensor operators $\tilde{T}^{(a''b'')t=k+1,q}_{y''i''i''_z}$ in terms of non-orthonormal tensor operators, which correspond to coupled states $|\eta_{+,\tilde{j}'',+})_q$, can be expressed as overlaps of elementary reduced matrix elements

$$(\tilde{T}^k|\eta^{+,\tilde{j}'',+})_q = \sum_{i'+z'=i+z-v} (\eta^{+,\tilde{j}'',+}|\zeta^{i'+z'})_q \langle aby\, i' + i''_0||\tilde{T}^{(a''b'')t=k+1,q}_{y''_0 i''_0}||a'b'y'i'\rangle_q \tag{3.5a}$$

$$= \sum_{i'+z'=i+z-v} \langle aby\, i' + i''_0||T^{k,k,t=k+1,q}_{k,k/2}||ab, y-k, i' + i''_0 - \tfrac{1}{2}k\rangle_q \times \langle ab, y-k, i' + i''_0 - \tfrac{1}{2}k||T^{a''-k,b''-k,t=1,q}_{y''_0-k,i''_0-k/2}||a'b'y'i'\rangle_q (\eta^{+,\tilde{j}'',+}|\zeta^{i'+z'})_q \tag{3.5b}$$

with inverse expansion coefficients (2.19). The first reduced matrix element in (3.5*b*) is expressed without a sum by means of (4.18) of [18], when the second special (stretched) reduced matrix element of the maximal null space tensor operator in (3.5*b*) is obtained as the next step. Equation (3.1) of [27] (together with (3.5) of [27] and the denominator functions (3.7) or (3.14) of [27]) for such extremal values of the parameters has fixed summation parameters $m' = i'$ and $j' = \frac{1}{2}(b - b'' - v - z' + i' + n_2)$ and gives

$$\langle aby\, i' + i''_0||T^{a''b''t=1,q}_{y''_0 i''_0}||a'b'y'i'\rangle_q = (a'b'y'i'; a''b''y''_0 i''_0||t=1; aby\, i' + i''_0)_q \tag{3.6a}$$

$$= \frac{([a+1][b+1][a+b+2][2i'+1]![i+z]![i-z]!)^{1/2}}{\mathcal{D}\binom{q,t=1}{a''b''}[a'b'; a\, b]([2i'+a''+1]![i'+z']![i'-z']!)^{1/2}} \times \left(\frac{[a+z+i+1]![b-z+i+1]![a'+z'-i']![b'-z'-i']!}{[a+z-i]![b-z-i]![a'+z'+i'+1]![b'-z'+i'+1]!}\right)^{1/2}$$
$$\times \sum_{n_1,n_2} \frac{(-1)^{n_1+n_2} q^{Q_2+a''n_1}[a+b+n_1+n_2+2]![b-z-i+n_2]!}{[n_1]![n_2]![a+n_1+1]![b+n_2+1]![a+b+n_1+2]!} \times \frac{[a-a''+v+n_1]![b'-a''+a+v+n_1+1]!}{[a+b+n_2+2]![a'+a''-a-v-n_1]![b'+b''-b+v-n_2]!}$$
$$\times \frac{[b-b''-v+n_2]![a'-b''+b-v+n_2+1]!}{[b-b''-v-z'-i'+n_2]![a+b-a''-b''+n_1+n_2+1]!} \tag{3.6b}$$

with $i = i' + i''_0$, $i + z = i' + z' + v$ and

$$Q_2 = \tfrac{1}{2}a''(a - a'' + v + z' - i') + \tfrac{1}{2}b''(b - b' - b'' - v + a' + 2z').$$

Expression (3.6*b*) resembles (3.7*b*) of [27] for the denominator function, but the second sum (over n_2) is of balanced ${}_5\phi_4$ type, instead of the unbalanced ${}_4\phi_3$. The sum over n_1 is indefinite for $a - a'' + v < 0$ when non-vanishing values of (3.6*b*) are impossible.

After inserting (3.6b) into (3.5b), the summation interval over $i' + z' = i + z - v$ is indefinite (the restricting q-factorials of the ${}_4\phi_3$ series cancel in the numerator and denominator). Therefore, it is reasonable to use the hook permutation symmetry [42] of the matrix elements under the substitution

$$\begin{aligned} &(a,b) \to (-a-b-3, a) \qquad (a',b') \to (-a'-b'-3, a') \\ &v \to a - a' + v \qquad z \to a + z + 1. \end{aligned} \tag{3.7}$$

Because of non-invariance under (3.7) for the initial definition of the denominator functions [27], the q-phase q^{Q_2} needs to be amended by $q^{-a''(a+b+2)}$. After inserting the last factor of (3.5b) in the new version, we find the following expression for the matrix element:

$$\begin{aligned} \langle aby\, i' + i_0'' || \tilde{T}^{(a''b'')t=k+1,q}_{y_0'' i_0''} || a'b'y'i' \rangle_q \\ &= (a'b'y'i'; a''-k, b''-k, y_0''-k, i_0'' - \tfrac{1}{2}k || t=1; a, b, y-k, i' + i_0'' - \tfrac{1}{2}k)_q \\ &\quad \times (a, b, y-k, i' + i_0'' - \tfrac{1}{2}k; k\,k\,k\,\tfrac{1}{2}k || t=k+1; aby\, i' + i_0'')_q \qquad (3.8a) \\ &= \frac{(-1)^{a-a'}}{\mathcal{D}\left({}^{q,t=1}_{a''-k,b''-k}\right)[a'b'; ab]\mathcal{D}\left({}^{q,t=k+1}_{k\,k}\right)[ab; ab]} \\ &\quad \times \left(\frac{[a+1][b+1][a+b+2][2i'+1]![i+z]![i-z]!}{[2i'+a''+1]![k]![i'+z']![i'-z']!} \right. \\ &\quad \left. \times \frac{[a+z+i+1]![b-z+i+1]![a'+z'-i']![b'-z'-i']!}{[a+z-i]![b-z-i]![a'+z'+i'+1]![b'-z'+i'+1]!} \right)^{1/2} \\ &\quad \times \sum_{n_1,n_2} \frac{(-1)^{n_1+n_2} q^{Q_3} [b-n_1]![a+b-n_1+1]![b-n_2]!}{[n_1]![n_2]![a''-b'+b-v-k-n_1]![b''+v-k-n_2]!} \\ &\quad \times \frac{[b+a''+b''-2k-n_1-n_2+1]![a'-b''-v+k+n_2]!}{[b-n_1-n_2]![a'+a''+b-v-k-n_1+2]![b'+b''-k+v-n_2+1]!} \\ &\quad \times \frac{[b''-k+i+z-n_2]!}{[a''+b-v-k-n_1+1]![i+z-n_2]![a+n_2+1]!} \qquad (3.8b) \end{aligned}$$

where

$$\begin{aligned} Q_3 = (a''-k)n_1 + \tfrac{1}{2}(a''-k)(-a-2b-4-a''+v+k+z'-i') \\ + \tfrac{1}{2}(b''-k)(b-b'-b''-v+a'+k+2z') + \tfrac{1}{2}k(k-3i+z-3). \end{aligned} \tag{3.9}$$

The sum over n_2 in (3.8b) corresponds to the balanced ${}_5\phi_4(q)$ basic hypergeometric series and forms the q-polynomial structure resembling (2.4), as well as the corresponding sum in (3.6b).

Now we return to the composition (3.5b) of (2.19) and (3.8b) for which the ${}_3\phi_2$ type sum over $i'+z'$ (balanced for $k=0$) may be rearranged in accordance with the symmetries and different versions of expressions for the Clebsch–Gordan coefficients of $\mathrm{u}_q(2)$ [43–45] or $\mathrm{u}_q(1,1)$, as follows,

$$\begin{aligned} \sum_x \frac{q^{-k(x-v)}[b''-k-n_2+x]![a+x+1]![b'+v-x]!}{[b'+v-s'-x]![x-n_2]![a+b''+s'+x+3]!} = \frac{[k]![b''-k]![a+b'+v+2]!}{[b'+b''+a+v+3]!} \\ \times \sum_s \frac{[a+n_2+1]![b'+b''-k+v-n_2+1]! q^{s(b'+b''+a+v+3)-k(b'-s')}}{[s]![b'+v-n_2-s'-s]![k-s]![a+n_2+s'+s+2]!} \\ \times \frac{[s'+s]!}{[b''-k+s'+s+1]!} \end{aligned} \tag{3.10}$$

in a such way that the new interval of summation is additionally restricted by k (hence it is finite for a tensor operator of fixed rank) and the sums over n_1 and n_2 turn into ${}_4\phi_3$ type sums, one of which is balanced. (However, its expression in terms of q-$6j$ coefficients is probably not helpful for further simplification.)

After replacing s by $s''-s'$, in analogy with (3.17) of [26] (in contrast with a failure to rearrange the denominator functions (3.7b) of [27] in the $u_q(3)$ case) we may use the standard ${}_2\phi_1$ summation formulae [36, 39] and rearrange this double sum in the composition (3.5b) of (2.19), (3.8b) and (3.10) as follows:

$$\sum_{n_1,n_2}\frac{(-1)^{n_1+n_2}q^{(a''-k)n_1}[b-n_1]![a+b-n_1+1]![b-n_2]!}{[n_1]![n_2]![b-n_1-n_2]![a''-b'+b-v-k-n_1]![b''+v-k-n_2]!}$$
$$\times\frac{[b+a''+b''-2k-n_1-n_2+1]!}{[a'+a''+b-v-k-n_1+2]![b'+v-n_2-s'']!}$$
$$\times\frac{[a'-b''-v+k+n_2]!}{[a''+b-v-k-n_1+1]![a+n_2+s''+2]!} \tag{3.11a}$$
$$=\sum_{n_1,n_2,s_1,s_2}\frac{(-1)^{n_1+n_2+s_1}q^{n_1n_2+s_1(b-n_1-n_2+1)}[b-s_1]!}{[s_1]![n_1-s_1]![n_2-s_1]![b'+v-n_2-s'']!}$$
$$\times\frac{q^{(a''-k)n_1}[a+b-n_1+1]!(-1)^{b''+v-k-n_2-s_2}[a'-s_2]!}{[a''-b'+b-v-k-n_1]![a+n_2+s''+2]!}$$
$$\times\frac{q^{-s_2(2k-a''-b''-b+n_1+n_2-1)-(b''+v-k-n_2)(a'+a''+b-v-k-n_1+2)}}{[s_2]![a'+a''+b-v-k-n_1-s_2+2]![b''+v-k-n_2-s_2]!} \tag{3.11b}$$
$$=\frac{[a+b'-a''+v+k+1]!}{[a+b'+v+2]!}\sum_{s_1,s_2}\frac{q^{s_1(s''+1)+s_2(b''-k+s''+1)}}{[s_1]![s_2]![b''+v-k-s_1-s_2]!}$$
$$\times\frac{q^{(b''+v-k)(b'+v-s'')-(a'-a-v)(a'+a''+b-v-k+2)}(-1)^{a''+b''-b'+b+s_1+s_2}}{[b'+v-s_1-s'']![a''-b'+b-v-k-s_1]!}$$
$$\times\frac{[b-s_1]![a'-s_2]![a''+b''-2k+v-s_1-s_2]!}{[a'+a''-a-v-k-s_2]![a+b''+v-k-s_2+s''+2]!} \tag{3.11c}$$
$$=\frac{(-1)^{a''+b''-b'+b}q^{(a-a'+v)(a+b'+v+2)+a''(b''+v-k)}[a+b'-a''+v+k+1]!}{[a+b'+v+2]![b'+v-s'']![a+b''+v-k+s''+2]!}$$
$$\times\frac{[b-b'-v+s'']![a''-k]!}{[a-a'+b''+v-k+s''+1]!}\sum_n\frac{(-1)^nq^{n(a+b'-a''+v+k+2)}}{[n]![a'+a''-a-v-k-n]!}$$
$$\times\frac{[a'-n]![b-b''-v+k+n]![a-a'+b''+v-k+s''+n+1]!}{[b''+v-k-n]![a-a'+v+n]![a-a'-a''+v+k+s''+n]!}. \tag{3.11d}$$

After we replace some q-factorials in (3.11c) as follows,

$$\frac{[a''+b''-2k+v-s_1-s_2]!}{[a''-b'+b-v-k-s_1]![a'+a''-a-v-k-s_2]![b''+v-k-s_1-s_2]!}$$
$$=\sum_n\{[a''-k]!q^{(a'+a''-a-v-k-s_2)(b''+v-k-s_1-s_2)-(n-s_2)(a''+b''-2k+v-s_1-s_2)}\}$$
$$\times\{[n-s_2]![a'+a''-a-v-k-n]!$$
$$\times[b''+v-k-s_1-n]![a-a'+v+n]!\}^{-1}$$

the summation over s_1 and s_2 is possible. The double sum in (3.11c) corresponds to a q-version of a special Kampe de Feriet series rearranged (up to an additional q-phase factor) in analogy with the relation

$$\sum_{r',s} \frac{(-1)^{r'+s} q^{r'(a+z-i+1)-s(i+z+1)} [2i'-s]![i-i'+i''+s]![a''-v+r']![a'-r']!}{[s]![i'+i''-i-s]![r']![v-r']![i'-z'-s-r']![i-i'+i''-v+s+r']!}$$
$$= \frac{[a''-v]![a+v+1]![i'-i''+i]!}{[i'+i''-i]![i-z]!}$$
$$\times \sum_n \frac{(-1)^n q^{n(a''-2i'')} [i'+z'+n]![a'-n]![i-z+v-n]!}{[n]![v-n]![i'-z'-n]![i+z-v+n]![a+v-n+1]!} \qquad (3.12)$$

(cf [46]) which follows from the expressions for the stretched q-9j coefficients (cf the $q = 1$ case of [38]) or (4.5) and (3.16) of [19] for the multiplicity-free isofactors $(a'b'y'i'; a''0y''i''||abyi)_q$ of $\mathrm{u}_q(3)$ with $b'' = 0$, $a + 2b = a' + 2b' + a''$.

Finally, the expansion coefficients (3.5a) may be expressed as follows:

$$(\tilde{T}^k|\eta^{+,\tilde{j}'',+})_q \equiv U_3 \left\{ \begin{matrix} (a'b') & (a''-k, \overset{1}{b''}-k) & (a\,b) \\ \overset{k+1}{(k\,k)} & (a\,b)^{+,\tilde{j}'',+} & (a''b'') \end{matrix} \right\}_q \qquad (3.13a)$$

$$= \frac{(-1)^{\tilde{j}''-\tilde{z}''}[a+b'-a''+v+k+1]!}{\mathcal{N}\binom{q,k}{a''b''}[a'b';ab]\nabla[\frac{1}{2}b',\frac{1}{2}b,\tilde{j}'']([\tilde{j}''+\tilde{z}'']!)^{1/2}}$$
$$\times \frac{H[a''b''\tilde{j}''\tilde{z}'']([2\tilde{j}''+1][\tilde{j}''-\tilde{z}'']!)^{1/2}}{([a+1]![a+b+2]![a']![a'+b'+1]![a'']![b'']![a''+b''+1]!)^{1/2}}$$
$$\times \sum_{n,s',s''} \frac{(-1)^{n+s'} q^{R+(s''-s')(b'+b''+a+v+3)+k(s'-b')+n(a+b'-a''+v+k+2)}}{[n]![b''+v-k-n]![a'+a''-a-v-k-n]![a-a'+v+n]!}$$
$$\times \frac{[a'-n]![b-b''-v+k+n]![\tilde{j}''+\tilde{z}''+s']![s'']![b-b'-v+s'']!}{[s']![\tilde{j}''-\tilde{z}''-s']![b-b'-v+s']![s''-s']![k+s'-s'']!}$$
$$\times \frac{[a-a'+b''+v-k+s''+n+1]![a+b''+v+s'+2]!}{[a-a'+b''+v-k+s''+1]![b'+v-s'']![b''-k+s''+1]!}$$
$$\times \frac{[b'+v-s']!}{[a-a'-a''+v+k+s''+n]![a+b''+v-k+s''+2]!} \qquad (3.13b)$$

with the same renormalization factor (3.3) and the q-phase

$$R = -\tilde{Q} + a''(b''+v-k) + \tfrac{1}{2}(b''-k)(a'-b'-b''+b-v+k) + \tfrac{1}{2}(a''-k)$$
$$\times(-a-2b-a''+v+k-4) + (a-a'+v)(a+b'+v+2)$$
$$+\tfrac{1}{2}k(k-2a''+v-3) - \tfrac{1}{4}(a-a''+2v)(a''+2b'')$$

with $\tilde{Q}$ as defined in (2.8). We see that expression (3.13b) is more symmetric than the expressions for the denominator functions [18, 27]: it is invariant under permutations of parameters of subarray (1.5b) of [27] which restrict the number of independent tensor operators and the summation interval for n. However, it is more convenient for $v \leqslant 0$, when this interval for n coincides with $\mathcal{M} - k - 1$. Specifically, for the canonical tensor operator with the minimal null space (which, in general, is not self-adjoint) and $v \leqslant 0$, n is fixed. The summation interval for s' is restricted by $\min[\tilde{j}'' - \tilde{z}'', \tilde{j}'' - \frac{1}{2}(b - b')]$ and the difference $s'' - s' \geqslant 0$ is restricted by k. Hence, they both do not exceed a''. Although the sum over s'' for $\mathcal{M} - k = 1$ and $v > 0$ resembles the Minton formula [36], they are not equivalent.

The condition $a - a'' + v + k \geqslant 0$ for the non-vanishing of special isofactors (2.4) or matrix elements (1.1) in (3.13*b*) is not necessary (contrary to the case of the denominator function (3.7*b*) of [27])—expression (3.13*b*) is indefinite (with possible negative q-factorials in the numerator) only for both $a-a''+v+k < 0$ and $a'-b''-v+k < 0$, conditions implied by the null space of the operator $\tilde{T}^{(a''b'')t=k+1,q}_{y''_0 i''_0 i''_z}$ (cf array (1.3*b*) and subarray (1.5*b*) of [27]). For $v > 0$ and $a - a'' + v + k < 0$ or $a' - b'' - v + k < 0$ the use of the isofactor symmetry relations (4.2*a*) and (4.2*b*) of [27] (possibly with transition to the overlaps $(\tilde{T}^k|\eta^{-,\tilde{J}'',-})_q$) may be required.

The separate series of ${}_4\phi_3$ and ${}_5\phi_4$ type in (3.13*b*) are rather remote from balanced ones, but a considerable number (ten) of the triadic correlations between their parameters appear (resembling those which were used to rearrange (3.11*a*) into (3.11*c*)). Although for $k = 0$ the sum over $s'' = s'$ can be accomplished in terms of a balanced ${}_3\phi_2$ series, attempts to rearrange the $k \geqslant 1$ case was unsuccessful. Note that the overlap $(\tilde{T}^0|\eta^{+,\tilde{j}'',+})_q$ may also be derived as a single sum (in terms of very well-poised series) as a composition of the overlaps $(\eta_{\hat{J}',\uparrow,\uparrow}|\eta^{\tilde{I}',-,-})_q$ (with $\hat{J}' = i_{\rm m} - i''_{\rm m}$, see (4.4) of [18] and (3.8) of [27] for renormalization) and $(\eta_{\tilde{I}',-,-}|\eta^{+,\tilde{j}'',+})_q$ (written after applying the symmetry properties to (3.1*b*) of [18]).

Using our overlaps (3.13*b*) and (3.1*c*) or (5.18) of [18], the general reduced matrix elements of operator (1.3) can be expanded in terms of the general bi-orthogonal isofactors as follows:

$$\langle abyi||\tilde{T}^{(a''b'')t=k+1,q}_{y''i''}||a'b'y'i'\rangle_q$$

$$= \sum_{\tilde{j}''} (\tilde{T}^k|\eta^{+,\tilde{j}'',+})_q (a'b'y'i';a''b''y''i''||_{+,\tilde{j}'',+};abyi)_q \tag{3.14a}$$

$$= \sum_{\tilde{I}} (\tilde{T}^k|\eta_{-,+,\tilde{I}})_q (a'b'y'i';a''b''y''i''||^{-,+,\tilde{I}};abyi)_q. \tag{3.14b}$$

The general overlap of the coupled tensor operators (1.3) may be expressed in terms of the auxillary overlaps as follows:

$$(\tilde{T}^k|\tilde{T}^{k'})_q = \sum_{y'i'y''i''} \langle abyi||\tilde{T}^{(a''b'')t=k+1,q}_{y''i''}||a'b'y'i'\rangle_q \langle abyi||\tilde{T}^{(a''b'')t'=k'+1,q}_{y''i''}||a'b'y'i'\rangle_q \tag{3.15a}$$

$$= \sum_{i',i,y-y'=y''} \frac{q^{-3y'} d_3[a''b''][2i+1]}{d_3[ab][2i''+1]} \langle abyi||\tilde{T}^{(a''b'')t=k+1,q}_{y''i''}||a'b'y'i'\rangle_q$$

$$\times \langle abyi||\tilde{T}^{(a''b'')t'=k'+1,q}_{y''i''}||a'b'y'i'\rangle_q \tag{3.15b}$$

$$= \sum_{\tilde{j}'',\tilde{J}''} (\tilde{T}^k|\eta^{+,\tilde{j}'',+})_q (\eta_{+,\tilde{j}'',+}|\eta_{+,\tilde{J}'',+})_q (\eta^{+,\tilde{J}'',+}|\tilde{T}^{k'})_q \tag{3.15c}$$

where the auxiliary triangular overlap matrix $(\eta_{+,\tilde{j}'',+}|\eta_{+,\tilde{J}'',+})_q$ may be expressed by means of (3.7) of [18] using the symmetry properties [27] of the boundary isofactors. (Recall that (3.15*a*) may be used for numerical orthonormalization and (3.15*b*) was only used effectively for overlaps of self-adjoined canonical tensor operators—see (4.14)—(4.15) of [18].) Nevertheless, the bilinear combination of $u_q(3)$ canonical recoupling coefficients and overlaps of the coupled tensor operators may be expressed more simply in terms of the following overlaps:

$$\sum_{t>\max(k,k')} U_3 \left\{ \begin{matrix} (a'b') & (a''-k, \overset{1}{b''}-k) & (a\,b) \\ \overset{k+1}{(k\,k)} & (a\,b) & \overset{t}{(a''b'')} \end{matrix} \right\}_q$$

$$\times U_3 \left\{ \begin{matrix} (a'b') & (a''-k', \overset{1}{b''}-k') & (a\,b) \\ \overset{k'+1}{(k'k')} & (a\,b) & \overset{t}{(a''b'')} \end{matrix} \right\}_q \tag{3.16a}$$

$$= (\tilde{T}^k|\tilde{T}^{k'})_q = \sum_{\tilde{j}'',\tilde{I}} (\tilde{T}^k|\eta^{+,\tilde{j}'',+})_q (\eta_{+,\tilde{j}'',+}|\eta^{-,+,\tilde{I}})_q (\eta_{-,+,\tilde{I}}|\tilde{T}^{k'})_q \tag{3.16b}$$

with the auxiliary overlap matrix

$$(\eta_{+,\tilde{j}'',+}|\eta^{-,+,\tilde{I}})_q = q^{Q_1(baa'b'b''a'';\tilde{j}'',-\tilde{z}'')-Q_1(b''a''b'a'ba;\tilde{I},-\tilde{z})-b'/2-a'}$$

$$\times(-1)^{b'-b+(b''-a')/2+\tilde{I}} \frac{\nabla[\frac{1}{2}b', \frac{1}{2}b, \tilde{j}'']H[a''b''\tilde{j}''\tilde{z}'']}{\nabla[\frac{1}{2}a', \frac{1}{2}b'', \tilde{I}]H[ab\tilde{I}\tilde{z}]}$$

$$\times\left(\frac{[a]![a+b+1]![a']![\tilde{j}''-\tilde{z}'']![\tilde{I}+\tilde{z}]![2\tilde{j}''+1]}{[a'']![a''+b''+1]![b']![\tilde{j}''+\tilde{z}'']![\tilde{I}-\tilde{z}]![b+1]}\right)^{1/2}$$

$$\times\frac{[2\tilde{I}+1][\tilde{I}+\tilde{j}''+\frac{1}{2}(a-a''+v)]!(a-a''+v|q)_{\tilde{I}-\tilde{j}''+(a''-a-v)/2}}{[\tilde{I}-\tilde{j}''+\frac{1}{2}(a''-a-v)]![\tilde{I}+\tilde{j}''+\frac{1}{2}(a''-a-v)+1]!} \tag{3.17}$$

expressed by means of (3.1b) of [18] using the symmetry properties and $(\eta_{-,+,\tilde{I}}|\tilde{T}^{k'})_q$ presented as (3.1c). The sum over $\tilde{j}''$ in (3.16b) is equivalent to the very well-poised ${}_8\phi_7$ series which may be transformed into a balanced ${}_4\phi_3$ series (cf (2.5.1) of [36], or (6.10) of [47]) but is not equivalent to a q-6j coefficient. Note that the overlap

$$(\tilde{T}^0|\tilde{T}^{k'})_q = \delta_{k',0} \tag{3.18}$$

is trivial, since $\tilde{T}^{(a''b'')t=1,q}_{y''i''i''_z}$ coincides with the unit canonical operator $T^{(a''b'')t=1,q}_{y''i''i''_z}$ which is orthogonal to each $\tilde{T}^{(a''b'')t>1,q}_{y''i''i''_z}$.

Finally, a solution of the system of equations (3.16a) and (3.16b), beginning from $k=\mathcal{M}-1$, allows us to orthonormalize the operators $\tilde{T}^{(a''b'')t=k+1,q}_{y''i''i''_z}$ and to expand their reduced matrix elements (using Gram determinants, cf [22]) either in terms of the bi-orthogonal isofactors with subscript $+, \tilde{j}'', +$ or in terms of the isofactors with superscript $-, +, \tilde{I}$, as well as in terms of the isofactors with superscript $+, \tilde{j}'', +$, which corresponds to the solution of the seed problem in [24].

4. The 'seed' isofactors of the minimal null space case

Let us consider separately the overlap matrices and 'seed' isofactors for the canonical tensor operators with the minimal null space, i.e. for $k=\mathcal{M}-1$. The expression of expansion coefficients (3.13b) with $k=\mathcal{M}-1$ may be simplified considerably for $v\leqslant 0$. Particularly, for $k=b''+v$ we obtained

$$(\tilde{T}^{b''+v}|\eta^{+,\tilde{j}'',+})_q = \frac{(-1)^{b''-\tilde{z}''-\tilde{j}''} q^R\, H[a''b''\tilde{j}''\tilde{z}'']\,\nabla[\frac{1}{2}b',\,\frac{1}{2}b,\,\tilde{j}'']}{\mathcal{N}(\substack{q,b''+v\\a''b''})[a'b';ab]} \times \frac{[b]![a'+b-v+1]!}{[-v]![a-a'+v]![b'-b+v]![b'+1]![a+b'+v+2]!} \times\left(\frac{[a']![2\tilde{j}''+1][\tilde{j}''-\tilde{z}'']!}{[a+1]![a+b+2]![a'+b'+1]![a'']![b'']![a''+b''+1]![\tilde{j}''+\tilde{z}'']!}\right)^{1/2} \times\sum_x \frac{(-1)^x q^{(b''+v)(a+v+2)-x(a+2)}[b''-x]![a+b'+v+x+2]!}{[x]![b''-\tilde{z}''-\tilde{j}''-x]![b''-\tilde{z}''+\tilde{j}''-x+1]![b-b''-v+x]!}. \tag{4.1}$$

For this purpose, in analogy with the transformations of (3.11a)–(3.11c) (cf also [39]), we rearranged the sum over n, s', s'' of (3.13b) as follows:

$$\frac{[a']![b]!}{[a-a'+v]![b'-b+v]!}\sum_{s',s_1'',s_2'',s_0''}\frac{(-1)^{s'}q^{(s_1''-s')(b'+b''+a+v+3)+(b''+v)(s'-b')}}{[s']![\tilde{j}''-\tilde{z}''-s']![b-b'-v+s']![s_2''-s']!} \times\frac{[\tilde{j}''+\tilde{z}''+s']![b'+v-s']![a+b''+v+s'+2]![s_2'']!}{[b''+v+s'-s_1'']![b'+v-s_1'']![a+s_1''+2]![-v+s_2''+1]!} \times\frac{(-1)^{s_0''-s_2''}q^{(v+s')(s_1''-s_2'')-s_0''(s_1''-s_2''-1)-s_1''}}{[s_1''-s_0'']![s_0''-s_2'']!}. \tag{4.2}$$

Summation over s_0'' gives $\delta_{s_1'',s_2''}$ and leads to a partial case of (3.13b). Otherwise, the double sum appears after summation over s_1'', s_2''. After the substitution of s_0'' by $b''+v+s'-x$, the sum over s' turns into a balanced one and, finally, we obtain (4.1). Similarly, the sum over n, s', s'' in (3.13b) may be rearranged for $k=a''-b'+b-v$, or for $k=a'+a''-a-v$, or for $k=a''$, respectively. It can also be derived (with the exception of a fixed n dependent factor $(-1)^n q^{n(a+b'-a''+v+k+2)}$) after the Regge-type substitutions

$$a'\leftrightarrow a+v \qquad b'\leftrightarrow b' \qquad a''\leftrightarrow a'+a''-a-v$$
$$b''\leftrightarrow a-a'+b''+v \qquad a\leftrightarrow a'-v \qquad b\leftrightarrow b \tag{4.3a}$$

or

$$a'\leftrightarrow a' \qquad b'\leftrightarrow b-v \qquad a''\leftrightarrow a''-b'+b-v$$
$$b''\leftrightarrow b'+b''-b+v \qquad a\leftrightarrow a \qquad b\leftrightarrow b'+v \tag{4.3b}$$

or

$$a'\leftrightarrow a+v \qquad b'\leftrightarrow b-v \qquad a''\leftrightarrow b''+v$$
$$b''\leftrightarrow a''-v \qquad a\leftrightarrow a'-v \qquad b\leftrightarrow b'+v \tag{4.3c}$$

of

$$([a+1]![a+b+2]![a']![a'+b'+1]![a'']![b'']![a''+b''+1]!)^{1/2} \times q^{-R}\mathcal{N}(\substack{q,k_{\max}\\a''b''})[a'b';ab](\tilde{T}^{k_{\max}}|\eta^{+,\tilde{j}'',+})_q$$

respectively, which correspond to the transpositions of the array (1.3b) of [27] or (2.8b) of [18].

Now let us consider the overlap matrices and 'seed' isofactors (3.1c) for the canonical tensor operators with the minimal null space. When $b-b'-v\leqslant 0$ in this extremal case the summation parameter j' in (3.1c) is fixed and the 'braided' sum (4.11) of [18] (depending on three parameters) may be used for summation over j. Particularly, for $k = b'' + v$ the asymmetric 'seed' isofactors (3.1c) may be expressed as follows:

$$(\tilde{T}^{b''+v}|\eta_{-,+,\tilde{I}})_q = q^{Q_6-(a'-v)(a'-v+2)/4+(a+b-b')(a+b-b'+2)/4+(b'+1)(b'-b+v)+3\tilde{I}(\tilde{I}+1)/2}$$
$$\times \frac{(-1)^{a-a''+\tilde{I}-\tilde{z}}([a+1][b+1][a+b+2][b']![a'+b'+1]!)^{1/2}}{\mathcal{N}\left(\begin{smallmatrix} q,b''+v \\ a''b'' \end{smallmatrix}\right)[a'b';ab]\nabla[\frac{1}{2}a',\frac{1}{2}b'',\tilde{I}]([b'']!)^{1/2}[\tilde{I}+\tilde{z}]![-v]!}$$
$$\times \frac{\Gamma[ab\tilde{I}\tilde{z}]\,[\tilde{I}-\tilde{z}]!\,[a']![b]![a'+b-v+1]!}{[a-a'+v]![b'-b+v]![a+1]![b'+1]![a+b'+v+2]!}. \qquad (4.4)$$

Combining (4.4) and (3.17), together with the 'braided' sum of the type (5.5) of [18] with

$$p_1 = \tfrac{1}{2}(a'-b'')-1 \qquad p_2 = \tfrac{1}{2}(a''-a-v)-\tilde{j}''-1$$
$$p_3 = \tfrac{1}{2}(a''-a-v)+\tilde{j}'' \qquad p_4 = \tfrac{1}{2}(a'+b'') \qquad p_5 = b-\tilde{z}$$

gives the overlaps correlated with the symmetric 'seed' isofactors and expressed in terms of the ${}_3\phi_2$ type basic hypergeometric series as follows:

$$(\tilde{T}^{b''+v}|\eta_{+,\tilde{j}'',+})_q = \sum_{\tilde{I}}(\tilde{T}^{b''+v}|\eta_{-,+,\tilde{I}})_q(\eta_{+,\tilde{j}'',+}|\eta^{-,+,\tilde{I}})_q \qquad (4.5a)$$
$$= q^{Q_6-(a'-v)(a'-v+2)/4+(a+b-b')(a+b-b'+2)/4+(b'+1)(b'-b+v)}$$
$$\times \frac{q^{(a'-b'')(a'-b''-2)/4+a'b''-b'/2-Q_1(a'b'a''b''ab;0\tilde{z})}}{[-v]![a-a'+v]![b'-b+v]![a+1]![b'+1]!}$$
$$\times \frac{\nabla[\frac{1}{2}b',\frac{1}{2}b,\tilde{j}'']H[a''b''\tilde{j}''\tilde{z}''][a']![b]!}{[a+b'+v+2]!\mathcal{N}\left(\begin{smallmatrix} q,b''+v \\ a''b'' \end{smallmatrix}\right)[a'b';ab]}$$
$$\times \left(\frac{[a+1]![a+b+2]![a']![a'+b'+1]![2\tilde{j}''+1]\,[\tilde{j}''+\tilde{z}'']!}{[a'']![b'']![a''+b''+1]![\tilde{j}''-\tilde{z}'']!}\right)^{1/2}$$
$$\times \sum_u \times\{q^{Q_1(b'a'aba''b'';\tilde{j}''\tilde{z}'')-u(a'+b-v+1)}[a'+a''-a-v-u]!\}$$
$$\times\{[u]![a'-u]![b''-\tilde{z}''-\tilde{j}''-u]![b-b''-v+u]!$$
$$\times[b''-\tilde{z}''+\tilde{j}''-u+1]!\}.^{-1} \qquad (4.5b)$$

Now extremal canonical seed isofactors for $t_{\max} = b'' + v$ may be written as follows:

$$(a'b'y_0'i_0'a''b''\overline{y}_0''\overline{i}_0''||t_{\max};ab\tilde{y}\tilde{I})_q$$
$$= (\tilde{T}^{t_{\max}}|\eta_{-,+,\tilde{I}})_q\left(\sum_{\tilde{j}''}(\tilde{T}^{t_{\max}}|\eta^{+,\tilde{j}'',+})_q(\tilde{T}^{t_{\max}}|\eta_{+,\tilde{j}'',+})_q\right)^{-1/2} \qquad (4.6)$$

$$(a'b'\overline{y}_0'\overline{i}_0'a''b''\tilde{y}''\tilde{j}''||t_{\max};ab\overline{y}_0\overline{i}_0)_q$$
$$= (\tilde{T}^{t_{\max}}|\eta_{+,\tilde{j}'',+})_q\left(\sum_{\tilde{j}''}(\tilde{T}^{t_{\max}}|\eta^{+,\tilde{j}'',+})_q(\tilde{T}^{t_{\max}}|\eta_{+,\tilde{j}'',+})_q\right)^{-1/2}. \qquad (4.7)$$

In both equations (4.6) and (4.7), the renormalization factors $\mathcal{N}\left(\begin{smallmatrix} q,t \\ a''b'' \end{smallmatrix}\right)[a'b';ab]$ cancel, as well as some other elementary factors. The sum over $\tilde{j}''$ in the new denominator function of (4.6) and (4.7) may be rearranged as follows:

$$\begin{aligned}
\boldsymbol{D}^2\left(\begin{smallmatrix} q,t=b''+v+1 \\ a''b'' \end{smallmatrix}\right)[a'b';ab] &= \frac{[a+1]!}{[a'+b-v+1]!}\sum_{\tilde{j}'',x,u}\frac{(-1)^{b''-\tilde{z}''-\tilde{j}''-x}q^{-x(a+2)}}{[x]![u]![b''-\tilde{z}''-\tilde{j}''-x]!} \\
&\quad\times\frac{[b''-x]![a+b'+v+x+2]![2\tilde{j}''+1]\nabla^2[\frac{1}{2}b',\frac{1}{2}b,\tilde{j}'']}{[b''-\tilde{z}''+\tilde{j}''-x+1]![b-b''-v+x]![a'-u]!} \\
&\quad\times\frac{H^2[a''b''\tilde{j}''\tilde{z}'']q^{-u(a'+b-v+1)}[a'+a''-a-v-u]!}{[b''-\tilde{z}''-\tilde{j}''-u]![b-b''-v+u]![b''-\tilde{z}''+\tilde{j}''-u+1]!} \qquad (4.8a) \\
&= \frac{[a''-v+1]![a+1]![b'+1]![a''+b-v+2]!}{[a'+b-v+1]!} \\
&\quad\times\sum_{s,x,u}\frac{[b-b'+a''-v-s]!}{[s]![a''+b-v+2-s]!} \\
&\quad\times\frac{(-1)^{b''+v-s-x}[b-s]![b'+b''-b+v+s+1]!q^{-x(a+2)-u(a'+b-v+1)}}{[x+u+s-b''-v]![b''+v-x-s]![b-b''-v+x]![b''+v-u-s]!} \\
&\quad\times\frac{[b''-x]![a+b'+v+x+2]![a'+a''-a-v-u]!}{[b-b''-v+u]![a'-u]![a'+a''+b''-a-v-x-u+1]!} \qquad (4.8b) \\
&= \frac{[a''-v+1]![a+1]![b'+1]![a''+b-v+2]![a'+a''-v+2]!}{[a'-b''+b-v]![a'+b-v+1]!} \\
&\quad\times\sum_{s,n}\frac{q^{n(b+b'+2)-(b''+v-s)(a'+b'+a+b+4)}[b-b'+a''-v-s]![b''-n]!}{[s]![b-s]![a''+b-v+2-s]![n]![b''+v-s-n]!} \\
&\quad\times\frac{[a'-b''+b-v+n]![b'-b+v+s+n]![a+b'+v+n+2]!}{[a'-b''-v+s+n]![a-b+b'+v+s+n+2]!}. \qquad (4.8c)
\end{aligned}$$

We rearranged the very well-poised ${}_8\phi_7$ series of (4.8*a*) into the balanced ${}_4\phi_3$ series in (4.8*b*) (related to a q-6j coefficient) using Watson's formula (2.5.1) of [36] as presented by (6.10) of [47] with parameters

$$\begin{aligned}
&a \to b'-b+1 \qquad b \to a''-v+2 \qquad c \to b'+b''-b+v+2 \\
&d \to b'+2 \qquad e \to u-b''-v \qquad N \to b''+v-x \qquad s \to \tfrac{1}{2}(b-b')+\tilde{j}''.
\end{aligned}$$

Further we replace some factors in (4.8*b*) as follows,

$$\begin{aligned}
&\frac{[b''-x]![a'+a''-a-v-u]![b'+b''-b+v+s+1]!}{[x+u+s-b''-v]![a'+a''+b''-a-v-x-u+1]!} \\
&\qquad = \sum_n q^{(b''-x+1)(n+u+s-b''-v)-(x-n)(a'+a''-a-v-u+1)} \\
&\qquad\quad\times\frac{[b''-n]![b'-b+v+s+n]!}{[x-n]![n+u+s-b''-v]!} \qquad (4.9)
\end{aligned}$$

and after summation over x and u obtain (4.8*c*), again rearranging a q-version of a special Kampe de Feriet series. An additional factor

$$\frac{[a+1]!}{[a'+b-v+1]!}$$

which is included in definition of (4.8*a*) is correlated with the Regge-type symmetry of seed isofactors and ensures a polynomial structure of the new denominator function (4.8*c*) for $q = 1$, in a definite analogy with the denominator polynomial as presented by (5.6) of [23]. Since both summation parameters are restricted by the same multiplicity parameter (contrary to the denominator function of the maximal null space case $\mathcal{D}\binom{q,t=1}{a''b''}[a'b';ab]$ [18, 27]), and all the separate terms are positive, our result may be useful for an extension of the generic SU(3) denominator function [23] to the $\mathrm{u}_q(3)$ canonical tensor operator case. Note that the definition of $\boldsymbol{D}\binom{q,t=b''+v+1}{a''b''}[a'b';ab]$ by (4.8c) in the self-adjoint case is not strictly correlated with (4.15b) of [18].

In a manner similar to (4.4) and (4.5b), we may derive the overlaps $(\tilde{T}^{a''-b'+b-v}|\eta_{-,+,\tilde{I}})_q$ and

$$(\tilde{T}^{a''-b'+b-v}|\eta_{+,\tilde{j}'',+})_q = q^{Q_6+(b'-b-a'+v)(b'-b-a'+v-2)/4+(a+v)(a-a'+b''+v+1)}$$
$$\times\frac{q^{(b'+1)(b'-b+v)+(a'-b'')(a'-b''-2)/4-a(a+2)/4-a'-b'/2-Q_1(a'b'a''b''ab;0\tilde{z})}}{[-v]![a'-a-v]![b'-b+v]![a'-v+1]![b'+1]!}$$
$$\times q^{Q_1(b'a'aba''b'';\tilde{j}''\tilde{z}'')}\frac{\nabla[\frac{1}{2}b',\frac{1}{2}b,\tilde{j}'']H[a''b''\tilde{j}''\tilde{z}''][a+v]![b]!}{[a'+b'+2]!\mathcal{N}\binom{q,a''-b'+b-v}{a''b''}[a'b';ab]}$$
$$\times\left(\frac{[a+1]![a+b+2]![a']![a'+b'+1]![2\tilde{j}''+1][\tilde{j}''+\tilde{z}'']!}{[a'']![b'']![a''+b''+1]![\tilde{j}''-\tilde{z}'']!}\right)^{1/2}$$
$$\times\sum_u\frac{q^{-u(a+b+1)}[a''-u]!([a''+\tilde{z}''+\tilde{j}''-u+1]!)^{-1}}{[u]![a+v-u]![a''+\tilde{z}''-\tilde{j}''-u]![b'-a''+v+u]!} \tag{4.10}$$

for $v \leqslant 0$, $k = a'' - b' + b - v$ (which up to a definite extension are related to (4.4) and (4.5b) by the Regge-type substitution (4.3a)), as well as $(\tilde{T}^{b''}|\eta_{-,+,\tilde{I}})_q$ and $(\tilde{T}^{b''}|\eta_{+,\tilde{j}'',+})_q$ or $(\tilde{T}^{a-a'+b''+v}|\eta_{-,+,\tilde{I}})_q$ and $(\tilde{T}^{a-a'+b''+v}|\eta_{+,\tilde{j}'',+})_q$ for $v \geqslant 0$. In this and the previous case all states $|\eta_{-,+,\tilde{I}})_q$ are linearly independent.

Otherwise, for $b'-b+v < 0$ the states $|\eta_{-,+,\tilde{I}})_q$ include some superfluous states. Then for the maximal values of k the summation parameter j in (3.1c) is fixed and the 'braided' sum (5.5) of [18] (depending on five parameters) may be used for rearrangement of the sum over j'. Particularly, for $k = a''$ we take (5.5) of [18] with

$$p_1 = \tfrac{1}{2}(b-a-b'-v)-1 \qquad p_2 = -\tfrac{1}{2}(a+b+b'+v)-2$$
$$p_3 = \tfrac{1}{2}(b+a-b'-v) \qquad p_4 = \tfrac{1}{2}a''+\tilde{I} \qquad p_5 = \tfrac{1}{2}a''-\tilde{I}-1$$

and obtain

$$(\tilde{T}^{a''}|\eta_{-,+,\tilde{I}})_q = (-1)^{a''+(a'+b'')/2-\tilde{I}}q^{Q_6+a'(a'+2)/4+a''(\tilde{z}-a''/4-1/2)+\tilde{I}(\tilde{I}+1)/2}$$
$$\times\frac{\Gamma[ab\tilde{I}\tilde{z}]([a+1][b+1][a+b+2][b']![a'+b'+1]!)^{1/2}}{\mathcal{N}\binom{q,a''}{a''b''}[a'b';ab][a'']!([b'']!)^{1/2}[-v]![a'-a-v]!}$$
$$\times\frac{\nabla[\frac{1}{2}b'',\frac{1}{2}a',\tilde{I}][a+v]![b'+v]![a+b'+v+1]!}{[b-b'-v]![a'-v+1]![b-v+1]![a'+b+v+2]!}$$
$$\times\sum_u\frac{(-1)^u q^{-ua''}[\tilde{I}-\tilde{z}+u]![b-u]!}{[u]![\tilde{I}+\tilde{z}-u]![b'+v-u]![a+u+1]!}. \tag{4.11}$$

Since its direct expansion in terms of $(\eta_{+,\tilde{j}'',+}|\eta^{-,+,\tilde{I}})_q$ does not simplify matters, we first derive

$$
\begin{aligned}
(\tilde{T}^{a''}|\eta_{\tilde{I}',-,-})_q &= \sum_{\tilde{I}} (\tilde{T}^{a''}|\eta_{-,+,\tilde{I}})_q (\eta^{-,+,\tilde{I}}|\eta_{\tilde{I}',-,-})_q \\
&= q^{Q_6 + a'(a'+2)/4 + a''(\tilde{z} - a''/4 - 1) - (a-a'')(a-a''+2)/4 - b'' - Q_1(a'b'a''b''ab;0\tilde{z})} \\
&\quad \times \frac{(-1)^{a-a'} \Gamma[a'b'\tilde{I}'\tilde{z}']([b+1]![a+b+2]![2\tilde{I}'+1])^{1/2}}{\mathcal{N}\binom{q,a''}{a''b''}[a'b';ab]\nabla[\frac{1}{2}a'', \frac{1}{2}a, \tilde{I}']([a'']!)^{1/2}[-v]![a'-a-v]!} \\
&\quad \times \frac{q^{Q_1(abb''a''a'b';\tilde{I}'\tilde{z}') + \tilde{I}'(\tilde{I}'+1)}[a+v]![b'+v]![a+b'+v+1]!}{[b-b'-v]![a'-v+1]![b-v+1]![a'+b+v+2]!}
\end{aligned}
\tag{4.12a}
$$

using (3.1b) and summation formula (3.6) of [18] with

$$
p_1 = \tilde{z} - u - 1 \qquad p_2 = \tfrac{1}{2}(b' - b + v) - \tilde{I}' - 1
$$

$$
p_3 = \tfrac{1}{2}(b' - b + v) + \tilde{I}' \qquad p_4 = b - \tilde{z}
$$

as well as an elementary summation formula of ${}_2\phi_1$. Then in analogy with (4.5b), using the overlap coefficient

$$
\begin{aligned}
(\eta_{+,\tilde{j}'',+}|\eta^{\tilde{I}',-,-})_q &= q^{(a''-b')/2 + b'' - a' + Q_1(b'a'aba''b'';\tilde{j}''\tilde{z}'') - Q_1(abb''a''a'b';\tilde{I}'\tilde{z}')} \\
&\quad \times (-1)^{\tilde{I}' + \tilde{z}' + \tilde{j}'' - \tilde{z}''} \frac{H[a''b''\tilde{j}''\tilde{z}'']\nabla[\frac{1}{2}b, \frac{1}{2}b', \tilde{j}'']}{H[a'b'\tilde{I}'\tilde{z}']\nabla[\frac{1}{2}a, \frac{1}{2}a'', \tilde{I}']} \\
&\quad \times \left(\frac{[a+1]![a']![a'+b'+1]![\tilde{j}''+\tilde{z}'']![\tilde{I}'+\tilde{z}']![2\tilde{j}''+1][2\tilde{I}'+1]}{[b+1]![b'']![a''+b''+1]![\tilde{j}''-\tilde{z}'']![\tilde{I}'-\tilde{z}']!} \right)^{1/2} \\
&\quad \times \frac{[\tilde{I}'+\tilde{j}''+\frac{1}{2}(a'-b''-v)]!(a'-b''-v|q)_{\tilde{I}'-\tilde{j}''+(b''-a'+v)/2}}{[\tilde{I}'-\tilde{j}''+\frac{1}{2}(b''-a'+v)]![\tilde{I}'+\tilde{j}''+\frac{1}{2}(b''-a'+v)+1]!}
\end{aligned}
\tag{4.13}
$$

and (5.5) of [18] with

$$
p_1 = -\tilde{z}' - 1 \qquad p_2 = \tfrac{1}{2}(b'' - a' + v) - \tilde{j}'' - 1
$$

$$
p_3 = \tfrac{1}{2}(b'' - a' + v) + \tilde{j}'' \qquad p_4 = b' - \tilde{z}' \qquad p_5 = \tfrac{1}{2}(a + a'')
$$

we obtain

$$
\begin{aligned}
(\tilde{T}^{a''}|\eta_{+,\tilde{j}'',+})_q &= \sum_{\tilde{I}'} (\tilde{T}^{a''}|\eta_{\tilde{I}',-,-})_q (\eta_{+,\tilde{j}'',+}|\eta^{\tilde{I}',-,-})_q \\
&= \frac{(-1)^{a'-a-v}\, q^{Q_6 + a'(a'+2)/4 + a''(a+\tilde{z}-a''/4+1/2) - Q_1(a'b'a''b''ab;0\tilde{z})}}{[-v]![a'-a-v]![b-b'-v]![a'-v+1]![b-v+1]!} \\
&\quad \times \frac{\nabla[\frac{1}{2}b', \frac{1}{2}b, \tilde{j}'']H[a''b''\tilde{j}''\tilde{z}''][a+v]![b'+v]!}{[a'+b+v+2]!\mathcal{N}\binom{q,a''}{a''b''}[a'b';ab]} \\
&\quad \times \left(\frac{[a+1]![a+b+2]![a']![a'+b'+1]![2\tilde{j}''+1]\,[\tilde{j}''+\tilde{z}'']!}{[a'']![b'']![a''+b''+1]![\tilde{j}''-\tilde{z}'']!} \right)^{1/2} \\
&\quad \times \sum_u \frac{q^{-a'-b'/2 + Q_1(b'a'aba''b'';\tilde{j}''\tilde{z}'') - u(a+b'+v+1)}}{[u]![a+v-u]![a''+\tilde{z}''-\tilde{j}''-u]![b'-a''+v+u]!} \\
&\quad \times \frac{[b-b'+a''-v-u]!}{[a''+\tilde{z}''+\tilde{j}''-u+1]!}.
\end{aligned}
\tag{4.14}
$$

Similarly, for $k = a' + a'' - a - v$ we derive $(\tilde{T}^k|\eta_{-,+,\tilde{I}})_q$, $(\tilde{T}^k|\eta_{\tilde{I}',-,-})_q$ and, finally, obtain

$$(\tilde{T}^{a'+a''-a-v}|\eta_{+,\tilde{j}'',+})_q = \frac{q^{Q_6+a'(a'+2)/4+(a'+a''-a-v)(a'-v+\tilde{z}-a''/4+1/2)-Q_1(a'b'a''b''ab;0\tilde{z})}}{[-v]![a-a'+v]![b-b'-v]![b-v+1]!}$$
$$\times \frac{\nabla[\frac{1}{2}b', \frac{1}{2}b, \tilde{j}'']H[a''b''\tilde{j}''\tilde{z}''][a']![b'+v]!}{\mathcal{N}\binom{q,a'+a''-a-v}{a''b''}[a'b';ab]([a+1]![a+b+2]!)^{1/2}}$$
$$\times \left(\frac{[a']![a'+b'+1]![2\tilde{j}''+1]\,[\tilde{j}''+\tilde{z}'']}{[a'']![b'']![a''+b''+1]![\tilde{j}''-\tilde{z}'']!}\right)^{1/2}$$
$$\times \sum_u \frac{q^{-a'-b'/2+Q_1(b'a'aba''b'';\tilde{j}''\tilde{z}'')-u(a'+b'+1)}}{[u]![a'-u]![b''-\tilde{z}''-\tilde{j}''-u]![b-b''-v+u]!}$$
$$\times \frac{[b''+v-u]!}{[b''-\tilde{z}''+\tilde{j}''-u+1]!}. \tag{4.15}$$

Hence, we see that all the

$$\left(\frac{[a'']![b'']![a''+b''+1]!}{[a+1]![a+b+2]![a']![a'+b'+1]!}\right)^{1/2} q^{-R'}\mathcal{N}\binom{q,k_{\max}}{a''b''}[a'b';ab](\tilde{T}^{k_{\max}}|\eta_{+,\tilde{j}'',+})_q$$

are related to one another through similar substitutions (4.3*a*)–(4.3*c*), which should also be applied to the denominator function (4.8*c*). Finally, in analogy with the Regge-type symmetry of the boundary paracanonical SU(3) isofactors [31], we see that renormalized extremal (seed) isofactors

$$\frac{(a'b'\bar{y}_0'\bar{i}_0'a''b''\tilde{y}''\tilde{j}''||t_{\max};ab\bar{y}_0\bar{i}_0)_q}{([a+1]![a+b+2]![a']![a'+b'+1]!)^{1/2}} \tag{4.16}$$

are also related to one another (up to sign and q-phase factors) by substitutions (4.3*a*)–(4.3*c*), as well as through their compositions with the usual transposition of the isofactor parameters

$$(ab) \leftrightarrow (a'b') \qquad a'' \leftrightarrow b'' \qquad v \rightarrow -v. \tag{4.17}$$

The renormalized extremal (seed) isofactors of the maximal null space case

$$\frac{(a'b'y_0'i_0'a''b''\tilde{y}''\tilde{I}''||t=1;aby_0i_0)_q}{([b+1]![a+b+2]![b']![a'+b'+1]!)^{1/2}} \tag{4.18}$$

(as presented by (5.14) of [18]) are also invariant (up to some q-phase factors) under the same substitutions, as well as the seed isofactors with arbitrary t.

5. Concluding remarks

In this paper we have demonstrated the importance of the distinctive polynomial and q-polynomial properties of definite extremal reduced matrix elements of the SU(3) and $u_q(3)$ canonical tensor operators for explicit analytical construction of the orthonormal 'seed' and general coupling coefficients with extremal and arbitrary multiplicity label. The expansion problem of general canonical coupling coefficients in terms of general bi-orthogonal isofactors with the subscript-type multiplicity label is reduced to an expansion of extremal reduced matrix elements in terms of the generalized Wilson–Rahman rational bi-orthogonal functions as balanced ${}_4F_3(1)$ and ${}_4\phi_3(q)$ hypergeometric series, proportional to q-6j coefficients with an unusual distribution of the matrix indices. Recall that seed isofactors usually serve as coefficients for an expansion in terms of general bi-orthogonal isofactors with the superscript-type multiplicity label. Composition (3.16*b*) of the expansion coefficients of both classes as overlaps of the generalized Draayer–Akiyama construction presents an explicit alternative of overlap (2.22*b*) of [22] and hence a Gram–Schmidt process that leads more directly to explicit orthonormal canonical coupling coefficients. We hope that the derived explicit denominator function of the minimal null space case will permit one to predict the new version of the general denominator function for canonical SU(3) tensor operators with a possible extension to the $u_q(3)$ case.

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